Indian Statistical Institute, Bangalore
B. Math (Hons.) Third Year
Second Semester - Complex Analysis

Final Exam — Date: 23rd April 2025
Maximum marks: 50 — Duration: 3 hours

Let $U = \{z \in \mathbb{C} \mid |z| < 1\}$.

Section I: Answer any four and each question carries 6 marks

1. Prove Cauchy-Riemann equation for analytic functions.
2. Suppose $f \in H(U)$ with $|f(z)| \leq 1$ for all $z \in U$. Prove that there is a $r > 0$ such that $|f(z)| \leq |f(0)| + \frac{3}{2}|z|$ for all z with $|z| < r$.
3. Suppose (f_n) is a sequence in $H(\Omega)$ and $f\colon \Omega \to \mathbb{C}$ is a function such that $f_n \to f$ uniformly on compact subsets of Ω. Prove that $f \in H(\Omega)$ and $f_n' \to f'$ uniformly on compact subsets of Ω.
4. Suppose $f\colon U \to U$ is analytic and $f(0) = 0$. Prove that $|f(z)| \leq |z|$ for all $z \in U$ and $|f'(0)| \leq 1$. If $|f(z)| = |z|$ for some $z \in U \setminus \{0\}$ or $|f'(0)| = 1$, prove that $f(z) = \lambda z$ for all $z \in U$ for some constant λ.
5. If $\lambda > 1$, find the cardinality of $\{z \in \mathbb{C} \mid \lambda - z - e^{-z} = 0, \quad z + \overline{z} > 0\}$. Justify your answer.
6. Evaluate $\int_0^{\pi} \frac{dx}{\sqrt{5} + \cos x}$ by residue theorem.

Section II: Answer any two, each question carries 13 marks

1. (a) Prove that the set of zeros of a non-zero analytic function on a region Ω has no limit point in Ω *(Marks: 6)*.

   (b) Determine all entier functions f such that $\Re(f(z)) \neq 0$ for all $z \in \mathbb{C}$ or $\Re(f(z)) \neq \pm\Im(f(z))$ for all $z \in \mathbb{C}$.
2. (a) Find all bijective analytic functions from U onto U with analytic inverse.

   (b) If f has no essential singularity at $z = a$, prove that f has a removable singularity or a pole at $z = a$ *(Marks: 6)*.
3. (a) State and prove the residue theorem *(Marks: 7)*.

   (b) Find all $f \in H(\mathbb{C})$ for which $f_*\colon \mathbb{C} \setminus \{0\} \to \mathbb{C}$ defined by $f_*(z) = f(1/z)$ for all $z \in \mathbb{C} \setminus \{0\}$ has no essential singularity at $z = 0$.